COVARIANT REPRESENTATIONS OF SUBPRODUCT SYSTEMS: INVARIANT SUBSPACES AND CURVATURE

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ABSTRACT. Let $X = (X(n))_{n\in\mathbb{Z}_+}$ be a standard subproduct system of C^*-correspondences over a C^*-algebra $\mathcal{M}$. Let $T = (T_n)_{n\in\mathbb{Z}_+}$ be a pure completely contractive, covariant representation of X on a Hilbert space $\mathcal{H}$. If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, then $\mathcal{S}$ is invariant for T if and only if there exist a Hilbert space $\mathcal{D}$, a representation π of $\mathcal{M}$ on $\mathcal{D}$, and a partial isometry $\Pi : \mathcal{F}_X \bigotimes_\pi \mathcal{D} \to \mathcal{H}$ such that

$$\Pi(S_n(\zeta) \otimes I_{\mathcal{D}}) = T_n(\zeta)\Pi \qquad (\zeta \in X(n), n \in \mathbb{Z}_+),$$

and $\mathcal{S} = \operatorname{ran} \Pi$, or equivalently, $P_{\mathcal{S}} = \Pi\Pi^*$. This result leads us to a list of consequences including Beurling-Lax-Halmos type theorem and other general observations on wandering subspaces. We extend the notion of curvature for completely contractive, covariant representations and analyze it in terms of the above results.

1. INTRODUCTION

Initiated by Gelu Popescu in [18], noncommutative Poisson transforms, and subsequently the explicit and analytic construction of isometric dilations, have been proved to be an extremely powerful tools in studying the structure of commuting and noncommuting tuples of bounded linear operators on Hilbert spaces. This is also important in noncommutative domains (and subsequently, noncommutative varieties) classification problems in the operator algebras (see [20], [21], [22] and references therein).

In [2] Arveson used similar techniques to generalize Sz.-Nagy and Foias dilation theory for commuting tuple of row contractions. These techniques have also led to further recent development [9, 11, 13, 14, 15, 19, 26, 27] on the structure of bounded linear operators in more general settings. In particular, in [15] Muhly and Solel introduced Poisson kernel for completely contractive, covariant representations over W^*-correspondences. The notion of Poisson kernel for completely contractive, covariant representations over a subproduct system of W^*-correspondences was introduced and studied by Shalit and Solel in [26]. This approach was further investigated by Viselter [27] for the extension problem of completely contractive, covariant representations of subproduct systems to C^*-representations of Toeplitz algebras.

Covariant representations on subproduct systems are important since it is one of the refined theories in operator theory and operator algebras that provides a unified approach to study commuting as well as noncommuting tuples of operators on Hilbert spaces.

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The main purpose of this paper is to investigate a Beurling-Lax-Halmos type invariant subspace theorem in the sense of [24, 25], and the notion of curvature in the sense of Arveson [3], Popescu [19] and Muhly and Solel [13] for completely contractive, covariant representations of standard subproduct systems.

The plan of the paper is the following. In Section 2, we recall several basic results from [27] including the intertwining property of Poisson kernels. In Section 3 we obtain an invariant subspace theorem for pure completely contractive, covariant representations of standard subproduct systems. As an immediate application we derive a Beurling-Lax-Halmos type theorem. Our objective in Section 4 is to extend, several results on curvature of a contractive tuple by Popescu [19, 20], for completely contractive, covariant representations of subproduct systems. We first define the curvature for completely contractive, covariant representations of standard subproduct systems. This approach is based on the definition of curvature for a completely contractive, covariant representation over a W^*-correspondence due to Muhly and Solel [13]. The final section is composed of several results on wandering subspaces which are motivated from our invariant subspace theorem. This section generalizes [5, Section 5] on wandering subspaces for commuting tuple of bounded operators on Hilbert spaces.

2. Notations and prerequisites

In this section, we recall some definitions and properties about C^*-correspondences and subproduct systems (see [16], [7], [11], [26]).

Let $\mathcal{M}$ be a C^*-algebra and let E be a Hilbert $\mathcal{M}$-module. Let $\mathcal{L}(E)$ be the C^*-algebra of all adjointable operators on E. The module E is said to be a *C^*-correspondence over $\mathcal{M}$* if it has a left $\mathcal{M}$-module structure induced by a non-zero $*$-homomorphism $\phi : \mathcal{M} \to \mathcal{L}(E)$ in the following sense

$$a\xi := \phi(a)\xi \qquad (a \in \mathcal{M}, \xi \in E).$$

All such $*$-homomorphisms considered in this article are non-degenerate, which means, the closed linear span of $\phi(\mathcal{M})E$ equals E. If F is another C^*-correspondence over $\mathcal{M}$, then we get the notion of tensor product $F \bigotimes_\phi E$ (cf. [7]) which satisfy the following properties:

$$(\zeta_1 a) \otimes \xi_1 = \zeta_1 \otimes \phi(a)\xi_1,$$
$$\langle \zeta_1 \otimes \xi_1, \zeta_2 \otimes \xi_2 \rangle = \langle \xi_1, \phi(\langle \zeta_1, \zeta_2 \rangle)\xi_2 \rangle$$

for all $\zeta_1, \zeta_2 \in F$; $\xi_1, \xi_2 \in E$ and $a \in \mathcal{M}$.

Assume $\mathcal{M}$ to be a W^*-algebra and E is a Hilbert $\mathcal{M}$-module. If E is self-dual, then E is called a *Hilbert W^*-module* over $\mathcal{M}$. In this case, $\mathcal{L}(E)$ becomes a W^*-algebra (cf. [16]). A C^*-correspondence over $\mathcal{M}$ is called a *W^*-correspondence* if E is self-dual, and if the $*$-homomorphism $\phi : \mathcal{M} \to \mathcal{L}(E)$ is normal. When E and F are W^*-correspondences, then their tensor product $F \bigotimes_\phi E$ is the self-dual extension of the above tensor product construction.

Definition 2.1. *Let $\mathcal{M}$ be a C^*-algebra, $\mathcal{H}$ be a Hilbert space, and E be a C^*-correspondence over $\mathcal{M}$. Assume $\sigma : \mathcal{M} \to B(\mathcal{H})$ to be a representation and $T : E \to B(\mathcal{H})$ to be a linear map. The tuple (T, σ) is called a* covariant representation *of E on $\mathcal{H}$ if*

$$T(a\xi a') = \sigma(a)T(\xi)\sigma(a') \qquad (\xi \in E, a, a' \in \mathcal{M}).$$

In the W^*-set up, we additionally assume that σ is normal and that T is continuous with respect to the σ-topology of E (cf. [4]) and ultra weak topology on $B(\mathcal{H})$. The covariant representation is called completely contractive *if T is completely contractive. The covariant representation (T, σ) is called* isometric *if*

$$T(\xi)^* T(\zeta) = \sigma(\langle \xi, \zeta \rangle) \qquad (\xi, \zeta \in E).$$

The following important lemma is due to Muhly and Solel [11, Lemma 3.5]:

Lemma 2.2. *The map $(T, \sigma) \mapsto \widetilde{T}$ provides a bijection between the collection of all completely contractive, covariant representations (T, σ) of E on $\mathcal{H}$ and the collection of all contractive linear maps $\widetilde{T} : E \bigotimes_\sigma \mathcal{H} \to \mathcal{H}$ defined by*

$$\widetilde{T}(\xi \otimes h) := T(\xi)h \qquad (\xi \in E, h \in \mathcal{H}),$$

and such that $\widetilde{T}(\phi(a) \otimes I_{\mathcal{H}}) = \sigma(a)\widetilde{T}$, $a \in \mathcal{M}$. Moreover, $\widetilde{T}$ is isometry if and only if (T, σ) is isometric.

Example: Assume E to be a Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^n$. Any contractive tuple $(T_1, \dots, T_n)$ on a Hilbert space $\mathcal{H}$ can be realized as a completely contractive, covariant representation (T, σ) of E on $\mathcal{H}$ where $T(e_i) := T_i$ for each $1 \leq i \leq n$, and when the representation σ maps every complex number λ to the multiplication operator by λ.

Now we recall several definitions and results from [27] which are essential for our objective. We will use A^*-algebra, to denote either C^*-algebra or W^*-algebra, to avoid repetitions in statements. Similarly we also use A^*-module and A^*-correspondence.

Definition 2.3. *Let $\mathcal{M}$ to be an A^*-algebra and $X = (X(n))_{n \in \mathbb{Z}_+}$ be a sequence of A^*-correspondences over $\mathcal{M}$. Then X is said to be a* subproduct system *over $\mathcal{M}$ if $X(0) = \mathcal{M}$, and for each $n, m \in \mathbb{Z}_+$ there exist a coisometric, adjointable bimodule function*

$$U_{n,m} : X(n) \bigotimes X(m) \to X(n+m),$$

such that

(a) *the maps $U_{n,0}$ and $U_{0,n}$ are the right and the left actions of $\mathcal{M}$ on $X(n)$, respectively, that is,*

$$U_{n,0}(\zeta \otimes a) := \zeta a, \; U_{0,n}(a \otimes \zeta) := a\zeta \qquad (\zeta \in X(n), \; a \in \mathcal{M}, n \in \mathbb{Z}_+),$$

(b) *the following associativity property holds for all $n, m, l \in \mathbb{Z}_+$;*

$$U_{n+m,l}(U_{n,m} \otimes I_{X(l)}) = U_{n,m+l}(I_{X(n)} \otimes U_{m,l}).$$

If each coisometric maps are unitaries, then we say the family X is a product system.

Definition 2.4. *Let $\mathcal{M}$ be an A^*-algebra and let $X = (X(n))_{n \in \mathbb{Z}_+}$ be a subproduct system over $\mathcal{M}$. Assume $T = (T_n)_{n \in \mathbb{Z}_+}$ to be a family of linear transformations $T_n : X(n) \to B(\mathcal{H})$, and define $\sigma := T_0$. Then the family T is called a* completely contractive, covariant representation of X on $\mathcal{H}$ *if*

(i) *for every* $n \in \mathbb{Z}_+$*, the pair* (T_n, σ) *is a completely contractive, covariant representation of the* A^**-correspondence* $X(n)$ *on* $\mathcal{H}$*, and*

(ii) *for every* $n, m \in \mathbb{Z}_+,\ \zeta \in X(n)$ *and* $\eta \in X(m)$*,*

$$T_{n+m}(U_{n,m}(\zeta \otimes \eta)) = T_n(\zeta)T_m(\eta). \tag{2.1}$$

For $n \in \mathbb{Z}_+$ define the contractive linear map $\widetilde{T}_n : X(n) \bigotimes_\sigma \mathcal{H} \to \mathcal{H}$ as (see [11])

$$\widetilde{T}_n(\zeta \otimes h) := T_n(\zeta)h \qquad (\zeta \in X(n),\ h \in \mathcal{H}). \tag{2.2}$$

Thus we can replace (2.1) by

$$\widetilde{T}_{n+m}(U_{n,m} \otimes I_{\mathcal{H}}) = \widetilde{T}_n(I_{X(n)} \otimes \widetilde{T}_m).$$

Example: The Fock space $\mathcal{F}_X := \bigoplus_{n\in\mathbb{Z}_+} X(n)$ of a subproduct system $X = (X(n))_{n\in\mathbb{Z}_+}$ is an A^*-correspondence over $\mathcal{M}$. For each $n \in \mathbb{Z}_+$, we define a linear map $S_n^X : X(n) \to \mathcal{L}(\mathcal{F}_X)$ by

$$S_n^X(\zeta)\eta := U_{n,m}(\zeta \otimes \eta)$$

for every $m \in \mathbb{Z}_+,\ \zeta \in X(n)$ and $\eta \in X(m)$. When $n \neq 0$ we call each operator S_n^X a *creation operator* of $\mathcal{F}_X$, and the family $S^X := (S_n^X)_{n\in\mathbb{Z}_+}$ is called an X*-shift*. It is easy to verify that the family S^X is indeed a completely contractive, covariant representation of $\mathcal{F}_X$. From the Definition 2.3 it is easy to see that, for each $a \in \mathcal{M}$, the map $S_0^X(a) = \phi_\infty(a) : \mathcal{F}_X \to \mathcal{F}_X$ maps $(b, \zeta_1, \zeta_2, \ldots) \mapsto (ab, a\zeta_1, a\zeta_2, \ldots)$.

Let $\mathcal{M}$ to be an A^*-algebra, and let $X = (X(n))_{n\in\mathbb{Z}_+}$ be an A^*-correspondences over $\mathcal{M}$. Then X is said to be a *standard subproduct system* if $X(0) = \mathcal{M}$, and for any $n, m \in \mathbb{Z}_+$ the bimodule $X(n+m)$ is an orthogonally complementable sub-module of $X(n) \bigotimes X(m)$.

Let $X = (X(n))_{n\in\mathbb{Z}_+}$ be a standard subproduct system and $E := X(1)$. Then for each n, the bi-module $X(n)$ is an orthogonally complementable sub-module of $E^{\otimes n}$ (here $E^{\otimes 0} = \mathcal{M}$), and hence there exists an orthogonal projection $p_n \in \mathcal{L}(E^{\otimes n})$ of $E^{\otimes n}$ onto $X(n)$. We denote the orthogonal projection $\bigoplus_{n\in\mathbb{Z}_+} p_n$ of $\mathcal{F}_E$, the Fock space of the product system $E = (E^{\otimes n})_{n\in\mathbb{Z}_+}$ with trivial unitaries, onto $\mathcal{F}_X$ by P.

Note also that here the projections $(p_n)_{n\in\mathbb{Z}_+}$ are bimodule maps and

$$p_{n+m} = p_{n+m}(I_{E^{\otimes n}} \otimes p_m) = p_{n+m}(p_n \otimes I_{E^{\otimes m}}),$$

for all $n, m \in \mathbb{Z}_+$. This implies that if we define each $U_{n,m}$ to be the projection p_{n+m} restricted to $X(n) \bigotimes X(m)$, then every standard subproduct system becomes a subproduct system over $\mathcal{M}$. In this case (2.1) reduces to

$$T_{n+m}(p_{n+m}(\zeta \otimes \eta)) = T_n(\zeta)T_m(\eta) \text{ for all } \zeta \in E^{\otimes n} \text{ and } \eta \in E^{\otimes m},$$

and (2.2) becomes

$$\widetilde{T}_{n+m}(p_{n+m} \otimes I_{\mathcal{H}})|_{X(n) \bigotimes X(m) \bigotimes_\sigma \mathcal{H}} = \widetilde{T}_n(I_{X(n)} \otimes \widetilde{T}_m). \tag{2.3}$$

Taking adjoints on both the sides we obtain

$$\widetilde{T}^*_{n+m} = (I_{X(n)} \otimes \widetilde{T}^*_m)\widetilde{T}^*_n \qquad (n, m \in \mathbb{Z}_+). \tag{2.4}$$

Note that for the sake of convenience we ignored the embedding of $X(n+m)\bigotimes_\sigma \mathcal{H}$ into $X(n)\bigotimes X(m)\bigotimes_\sigma \mathcal{H}$ in the previous formula. We further deduce that

$$\widetilde{T}_{n+1}^* = (I_E \otimes \widetilde{T}_n^*)\widetilde{T}_1^* = (I_{X(n)} \otimes \widetilde{T}_1^*)\widetilde{T}_n^*, \tag{2.5}$$

and

$$\widetilde{T}_n^* = (I_{X(n-1)} \otimes \widetilde{T}_1^*)(I_{X(n-2)} \otimes \widetilde{T}_1^*)\dots(I_E \otimes \widetilde{T}_1^*)\widetilde{T}_1^*,$$

for all $n \in \mathbb{Z}_+$.

Example: If $X(n)$ is the n-fold symmetric tensor product of the Hilbert space $X(1)$, then $X = (X(n))_{n\in\mathbb{Z}_+}$ becomes a standard subproduct system of Hilbert spaces (cf. [26, Example 1.3]). Moreover, let $\{e_1, \dots, e_d\}$ be an orthonormal basis of $X(1)$. Then

$$T \leftrightarrow (T_1(e_1), T_1(e_2), \dots, T_1(e_d))$$

induces a bijection between the set of all completely contractive covariant representations T of X on a Hilbert space $\mathcal{H}$ onto the collection of all commuting row contractions $(T_1, \dots, T_d)$ on $\mathcal{H}$ (cf. Example 5.6, [26]).

Before proceeding to the notion of Poisson kernels, we make a few comments:

1. We use the symbol sot-lim for the limit with respect to the strong operator topology. From Equation 2.5 we infer that $\{\widetilde{T}_n\widetilde{T}_n^*\}_{n\in\mathbb{Z}_+}$ is a decreasing sequence of positive contractions, and thus $Q :=$ sot-$\lim\limits_{n\to\infty} \widetilde{T}_n\widetilde{T}_n^*$ exists. If $Q = 0$, then we say that the covariant representation T is *pure*. Note that T is pure if and only if sot-$\lim\limits_{n\to\infty} \widetilde{T}_n^* = 0$.
2. Let ψ be a representation of $\mathcal{M}$ on a Hilbert space $\mathcal{E}$. Then the induced covariant representation $S \otimes I_{\mathcal{E}} := (S_n(\cdot) \otimes I_{\mathcal{E}})_{n\in\mathbb{Z}_+}$ is pure, where each $S_n(\cdot) \otimes I_{\mathcal{E}}$ is an operator from $X(n)$ into $B(\mathcal{F}_X \bigotimes_\psi \mathcal{E})$.
3. It is proved in [26, Lemma 6.1] that every subproduct system is isomorphic to a standard subproduct system. Therefore it is enough to consider standard subproduct systems.

Let $T = (T_n)_{n\in\mathbb{Z}_+}$ be a completely contractive, covariant representation of a standard subproduct system $X = (X(n))_{n\in\mathbb{Z}_+}$. We denote the positive operator $(I_{\mathcal{H}} - \widetilde{T}_1\widetilde{T}_1^*)^{1/2} \in B(\mathcal{H})$ by $\triangle_*(T)$ and the defect space $\overline{\mathrm{Im}\,\triangle_*(T)}$ by $\mathcal{D}$. It is proved in [27, Proposition 2.9] that $\triangle_*(T) \in \sigma(\mathcal{M})'$. Therefore $\mathcal{D}$ reduces $\sigma(a)$ for each $a \in \mathcal{M}$. Thus using the reduced representation σ' we can form the tensor product of the Hilbert space $\mathcal{D}$ with $X(n)$ for each $n \in \mathbb{Z}_+$, and hence with $\mathcal{F}_X$. For simplicity we write σ instead of σ'. The *Poisson kernel* of T is the operator $K(T) : \mathcal{H} \to \mathcal{F}_X \bigotimes_\sigma \mathcal{D}$ defined by

$$K(T)h := \sum_{n\in\mathbb{Z}_+} (I_{X(n)} \otimes \triangle_*(T))\widetilde{T}_n^* h \qquad (h \in \mathcal{H}).$$

In the next proposition we recall the properties of the Poisson kernel from [27]:

Proposition 2.5. *Let $T = (T_n)_{n\in\mathbb{Z}_+}$ be a completely contractive, covariant representation of a standard subproduct system $X = (X(n))_{n\in\mathbb{Z}_+}$ over an A^*-algebra $\mathcal{M}$. Then*

$K(T)$ is a contraction and

$$K(T)^*(S_n(\zeta)\otimes I_{\mathcal{D}}) = T_n(\zeta)K(T)^* \qquad (n\in\mathbb{Z}_+,\ \zeta\in X(n)).$$

Moreover, $K(T)$ is an isometry if and only if T is pure.

Proof. For each $h\in\mathcal{H}$, from (2.5) and (2.3) it follows that

$$\begin{aligned}\sum_{n\in\mathbb{Z}_+}\|(I_{X(n)}\otimes\triangle_*(T))\widetilde{T}_n^*h\|^2 &= \sum_{n\in\mathbb{Z}_+}\langle\widetilde{T}_n(I_{X(n)}\otimes\triangle_*(T)^2)\widetilde{T}_n^*h,h\rangle\\ &= \sum_{n\in\mathbb{Z}_+}\langle\widetilde{T}_n(I_{X(n)}\otimes(I_{\mathcal{H}}-\widetilde{T}_1\widetilde{T}_1^*))\widetilde{T}_n^*h,h\rangle\\ &= \sum_{n\in\mathbb{Z}_+}\langle\widetilde{T}_n\widetilde{T}_n^*-\widetilde{T}_{n+1}\widetilde{T}_{n+1}^*h,h\rangle\\ &= \langle h,h\rangle-\lim_{n\to\infty}\langle\widetilde{T}_n\widetilde{T}_n^*h,h\rangle,\end{aligned}$$

here we also used $\widetilde{T}_0\widetilde{T}_0^*=I_{\mathcal{H}}$. So $K(T)$ is a well-defined contraction, and it is an isometry if T is pure. Now for each $n\in\mathbb{Z}_+$ and $z_n\in X(n)\bigotimes_\sigma\mathcal{D}$ we have

$$K(T)^*\left(\sum_{n\in\mathbb{Z}_+}z_n\right)=\sum_{n\in\mathbb{Z}_+}\widetilde{T}_n(I_{X(n)}\otimes\triangle_*(T))z_n.$$

Therefore for every $m\in\mathbb{Z}_+$, $\eta\in X(m)$ and $h\in\mathcal{D}$, (2.5) gives

$$\begin{aligned}K(T)^*(S_n(\zeta)\otimes I_{\mathcal{D}})(\eta\otimes h) &= K(T)^*(p_{n+m}(\zeta\otimes\eta)\otimes h)\\ &= \widetilde{T}_{n+m}(p_{n+m}(\zeta\otimes\eta)\otimes\triangle_*(T)h)\\ &= \widetilde{T}_n(\zeta\otimes\widetilde{T}_m(\eta\otimes\triangle_*(T)h))\\ &= T_n(\zeta)K(T)^*(\eta\otimes h).\quad\square\end{aligned}$$

3. Invariant subspaces of covariant representations

In this section we first introduce the notion of invariant subspaces for completely contractive, covariant representations and then in Theorem 3.1 we obtain a far reaching generalization of [24, Theorem 2.2].

Let $T=(T_n)_{n\in\mathbb{Z}_+}$ be a completely contractive, covariant representation of a standard subproduct system $X=(X(n))_{n\in\mathbb{Z}_+}$ over an A^*-algebra $\mathcal{M}$. A closed subspace $\mathcal{S}$ of $\mathcal{H}$ is called *invariant* for the covariant representation T if $\mathcal{S}$ is invariant for $\sigma(\mathcal{M})$ and if $\mathcal{S}$ is left invariant by each operator in the set $\{T_n(\zeta):\zeta\in X(n),n\in\mathbb{N}\}$.

Theorem 3.1. *Let $T=(T_n)_{n\in\mathbb{Z}_+}$ be a pure completely contractive, covariant representation of a standard subproduct system $X=(X(n))_{n\in\mathbb{Z}_+}$ over an A^*-algebra $\mathcal{M}$, and let $\mathcal{S}$ be a non-trivial closed subspace of $\mathcal{H}$. Then $\mathcal{S}$ is invariant for T if and only if there exist a Hilbert space $\mathcal{D}$, a representation π of $\mathcal{M}$ on $\mathcal{D}$, and a partial isometry $\Pi:\mathcal{F}_X\bigotimes_\pi\mathcal{D}\to\mathcal{H}$ such that $\mathcal{S}=ran\,\Pi$ and*

$$\Pi(S_n(\zeta)\otimes I_{\mathcal{D}})=T_n(\zeta)\Pi \qquad (\zeta\in X(n),\ n\in\mathbb{Z}_+).$$

Proof. Since $\mathcal{S}$ is invariant for $T=(T_n)_{n\in\mathbb{Z}_+}$, we get a covariant representation $(V_n := T_n|_{\mathcal{S}})_{n\in\mathbb{Z}_+}$ of the standard subproduct system $X=(X(n))_{n\in\mathbb{Z}_+}$ on $\mathcal{S}$. We denote V_0 by π. Now for each $n\in\mathbb{N}$, $s\in\mathcal{S}$, and $\zeta\in X(n)$,

$$\langle \zeta\otimes s,\zeta\otimes s\rangle = \langle s,\pi(\langle\zeta,\zeta\rangle)s\rangle = \langle s,\sigma(\langle\zeta,\zeta\rangle)s\rangle = \langle \zeta\otimes s,\zeta\otimes s\rangle,$$

yields an embedding j_n from $X(n)\bigotimes_{\pi}\mathcal{S}$ into $X(n)\bigotimes_{\sigma}\mathcal{H}$. Thus for each $n\in\mathbb{N}$, $j_nj_n^*$ is an orthogonal projection.
For each $n\in\mathbb{N}$, from the definition of the map $\widetilde{V}_n : X(n)\bigotimes_{\pi}\mathcal{S}\to\mathcal{S}$ it follows that

$$\widetilde{V}_n(\zeta\otimes s) = V_n(\zeta)s = T_n(\zeta)s = \widetilde{T}_n\circ j_n(\zeta\otimes s),$$

for all $\zeta\in X(n)$ and $s\in\mathcal{S}$. It also follows that

$$\langle \widetilde{V}_n\widetilde{V}_n^* s,s\rangle = \langle \widetilde{T}_n j_n j_n^*\widetilde{T}_n^* s,s\rangle \le \langle \widetilde{T}_n\widetilde{T}_n^* s,s\rangle,$$

for all $n\in\mathbb{N}$ and $s\in\mathcal{S}$. Hence the covariant representation V is pure as well as completely contractive.
Since the defect space $\mathcal{D}=\overline{\operatorname{Im}\triangle_*(V)}$ of the representation V is reducing for π, it follows from Proposition 2.5 that the Poisson kernel $K(V):\mathcal{S}\to\mathcal{F}_X\bigotimes_{\pi}\mathcal{D}$, defined by

$$K(V)(s) = \sum_{n\in\mathbb{Z}_+}(I_{X(n)}\otimes\triangle_*(V))\widetilde{V}_n^* s \qquad (s\in\mathcal{S}),$$

is an isometry and

$$K(V)^*(S_n(\zeta)\otimes I_{\mathcal{D}}) = V_n(\zeta)K(V)^*,$$

for all $n\in\mathbb{Z}_+$, and $\zeta\in X(n)$. Let $i_{\mathcal{S}}:\mathcal{S}\to\mathcal{H}$ be the inclusion map. Clearly $i_{\mathcal{S}}$ is an isometry and

$$i_{\mathcal{S}}T_n(\cdot)|_{\mathcal{S}} = T_n(\cdot)i_{\mathcal{S}}.$$

Therefore we get a map $\Pi:\mathcal{F}_X\bigotimes_{\pi}\mathcal{D}\to\mathcal{H}$ defined by $\Pi := i_{\mathcal{S}}K(V)^*$. Then

$$\Pi\Pi^* = i_{\mathcal{S}}K(V)^*(i_{\mathcal{S}}K(V)^*)^* = i_{\mathcal{S}}i_{\mathcal{S}}^* = P_{\mathcal{S}},$$

the projection on $\mathcal{S}$. Hence Π is a partial isometry and the range of Π is $\mathcal{S}$. From $i_{\mathcal{S}}V_n = i_{\mathcal{S}}T_n|_{\mathcal{S}} = T_n i_{\mathcal{S}}$ and the intertwining property of the Poisson kernel we deduce that

$$\Pi(S_n(\zeta)\otimes I_{\mathcal{D}}) = i_{\mathcal{S}}K(V)^*(S_n(\zeta)\otimes I_{\mathcal{D}}) = i_{\mathcal{S}}V_n(\zeta)K(V)^* = T_n(\zeta)\Pi.$$

Conversely, suppose that there exists a partial isometry $\Pi:\mathcal{F}_X\bigotimes_{\pi}\mathcal{D}\to\mathcal{H}$. Then $ran\Pi$ is a closed subspace of $\mathcal{H}$ and the intertwining relation for Π implies that $ran\Pi$ is a $T=(T_n)_{n\in\mathbb{Z}_+}$ invariant subspace of $\mathcal{H}$. □

Corollary 3.2. *Let $T=(T_n)_{n\in\mathbb{Z}_+}$ be a pure completely contractive, covariant representation of a standard subproduct system $X=(X(n))_{n\in\mathbb{Z}_+}$ over an A^*-algebra $\mathcal{M}$, and $\mathcal{S}$ be a non-trivial closed subspace of $\mathcal{H}$. Then $\mathcal{S}$ is invariant for T if and only if there exist a Hilbert space $\mathcal{D}$, a representation π of $\mathcal{M}$ on $\mathcal{D}$, and a bounded linear operator $\Pi:\mathcal{F}_X\bigotimes_{\pi}\mathcal{D}\to\mathcal{H}$ such that $P_{\mathcal{S}}=\Pi\Pi^*$, and*

$$\Pi(S_n(\zeta)\otimes I_{\mathcal{D}}) = T_n(\zeta)\Pi \qquad (\zeta\in X(n),\ n\in\mathbb{Z}_+).$$

Definition 3.3. *Let $X = (X(n))_{n\in\mathbb{Z}_+}$ be a standard subproduct system over an A^*-algebra $\mathcal{M}$. Assume ψ and π to be representations of $\mathcal{M}$ on Hilbert spaces $\mathcal{E}$ and $\mathcal{E}'$, respectively. A bounded operator $\Pi : \mathcal{F}_X \bigotimes_\pi \mathcal{E}' \to \mathcal{F}_X \bigotimes_\psi \mathcal{E}$ is called* multi-analytic *if it satisfies the following condition*

$$\Pi(S_n(\zeta) \otimes I_{\mathcal{E}'}) = (S_n(\zeta) \otimes I_{\mathcal{E}})\Pi \quad whenever\ \zeta \in X(n),\ n \in \mathbb{Z}_+.$$

Further we call it inner *if it is a partial isometry.*

As an application, we have the following Beurling-Lax-Halmos type theorem (cf. [20, Theorem 3.2]) which extends [17, Theorem 2.4] and [25, Corollary 4.5]:

Theorem 3.4. *Assume $X = (X(n))_{n\in\mathbb{Z}_+}$ to be a standard subproduct system over an A^*-algebra $\mathcal{M}$ and assume ψ to be a representation of $\mathcal{M}$ on a Hilbert space $\mathcal{E}$. Let $\mathcal{S}$ be a non-trivial closed subspace of the Hilbert space $\mathcal{F}_X \bigotimes_\psi \mathcal{E}$. Then $\mathcal{S}$ is invariant for $S \otimes I_{\mathcal{E}}$ if and only if there exist a Hilbert space $\mathcal{E}'$, a representation π of $\mathcal{M}$ on $\mathcal{E}'$, and an inner multi-analytic operator $\Pi : \mathcal{F}_X \bigotimes_\pi \mathcal{E}' \to \mathcal{F}_X \bigotimes_\psi \mathcal{E}$ such that $\mathcal{S}$ is the range of Π.*

Proof. Let $\mathcal{S}$ be an invariant subspace for $S \otimes I_{\mathcal{E}}$. By Theorem 3.1 we know that there exist a Hilbert space $\mathcal{E}'$, a representation π of $\mathcal{M}$ on $\mathcal{E}'$, and a partial isometry $\Pi : \mathcal{F}_X \bigotimes_\pi \mathcal{E}' \to \mathcal{F}_X \bigotimes_\psi \mathcal{E}$ such that $\mathcal{S} = \text{ran}\,\Pi$ and

$$\Pi(S_n(\zeta) \otimes I_{\mathcal{E}'}) = (S_n(\zeta) \otimes I_{\mathcal{E}})\Pi \qquad (\zeta \in X(n),\ n \in \mathbb{Z}_+).$$

For the reverse direction, if we start with a partial isometry $\Pi : \mathcal{F}_X \bigotimes_\pi \mathcal{E}' \to \mathcal{F}_X \bigotimes_\psi \mathcal{E}$, then $ran\Pi$ is a closed subspace of $\mathcal{F}_X \bigotimes_\psi \mathcal{E}$ and the intertwining relation for Π implies that $ran\Pi = \mathcal{S}$ is invariant for $S \otimes I_{\mathcal{E}}$. □

For Beurling type classification in the tensor algebras setting see also Muhly and Solel [12, Theorem 4.7].

4. Curvature

The notion of a curvature for commuting tuples of row contractions was introduced by Arveson [3]. This numerical invariant is an analogue of the Gauss-Bonnet-Chern formula from Riemannian geometry, and closely related to rank of Hilbert modules over polynomial algebras. It has since been further analyzed by Popescu [19] (see also [23] for recent results on a general class), Kribs [10] in the setting of noncommuting tuples of operatos, and by Muhly and Solel [13] in the setting of completely positive maps on C^* algebras of bounded linear operators.

The purpose of this section is to study curvature for a more general framework, namely, for completely contractive, covariant representations of subproduct systems.

We begin by recalling the definition of left dimension [6] for a W^*-correspondences E over a semifinite factor $\mathcal{M}$ (see Muhly and Solel, Definition 2.5, [13]).

Let $\mathcal{M}$ be a semifinite factor and τ be a faithful normal semifinite trace, and let $L^2(\mathcal{M})$ be the GNS construction for τ. Note that for each $a \in \mathcal{M}$ there exists a left multiplication operator, denoted by $\lambda(a)$, and a right multiplication operator, denoted by $\rho(a)$, on $L^2(\mathcal{M})$. Each unital, normal, $*$-representation $\sigma : \mathcal{M} \to B(\mathcal{H})$ defines a *left*

$\mathcal{M}$*-module* $\mathcal{H}$. This yields an $\mathcal{M}$-linear isometry $V : \mathcal{H} \to L^2(\mathcal{M}) \bigotimes l_2$. Here $\mathcal{M}$-linear means

$$V\sigma(a) = (\lambda(a) \otimes I_{l_2})V \qquad (a \in \mathcal{M}).$$

Moreover

$$V\sigma(\mathcal{M})'V^* = p(\lambda(\mathcal{M}) \otimes I_{l_2})'p \subseteq (\lambda(\mathcal{M}) \otimes I_{l_2})',$$

where $p := VV^* \in (\lambda(\mathcal{M}) \otimes I_{l_2})'$ is a projection. One can observe that $(\lambda(\mathcal{M}) \otimes I_{l_2})'$ equals the semifinite factor $\rho(\mathcal{M}) \bigotimes B(l_2)$ whose elements can be written as matrices of the form $(\rho(a_{ij}))$. For each positive element $x \in \sigma(\mathcal{M})'$, we express VxV^* in the form $(\rho(a_{ij}))$, and define

$$tr_{\sigma(\mathcal{M})'}(x) := \sum \tau(a_{ii}).$$

Note that $tr_{\sigma(\mathcal{M})'}$ is a faithful normal semifinite trace on $\sigma(\mathcal{M})'$. The *left dimension* of $\mathcal{H}$ is defined by

$$dim_l(\mathcal{H}) := tr_{\sigma(\mathcal{M})'}(p).$$

For each W^*-correspondence E, the Hilbert space $E \bigotimes_\sigma L^2(\mathcal{M})$ has a natural left $\mathcal{M}$-module structure. The left dimension of $E \bigotimes_\sigma L^2(\mathcal{M})$ will be denoted by $dim_l(E)$.

Now let $X = (X(n))_{n\in\mathbb{Z}_+}$ be a standard subproduct system of W^*-correspondences over a semifinite factor $\mathcal{M}$. Let $T = (T_n)_{n\in\mathbb{Z}_+}$ be a completely contractive, covariant representation of X on a Hilbert space $\mathcal{H}$. Define a contractive normal, completely positive map $\Theta_T : \sigma(\mathcal{M})' \to \sigma(\mathcal{M})'$ by

$$\Theta_T(a) := \widetilde{T}_1(I_E \otimes a)\widetilde{T}_1^* \qquad (a \in \sigma(\mathcal{M})').$$

It follows from (2.3), (2.4) and (2.5) that

$$\begin{aligned}\Theta_T^2(a) &= \Theta_T(\Theta_T(a)) = \widetilde{T}_1(I_E \otimes (\widetilde{T}_1(I_E \otimes a)\widetilde{T}_1^*))\widetilde{T}_1^* \\ &= \widetilde{T}_1(I_E \otimes \widetilde{T}_1)(I_{E^{\otimes 2}} \otimes a)(I_E \otimes \widetilde{T}_1^*)\widetilde{T}_1^* \\ &= \widetilde{T}_2(p_2 \otimes I_\mathcal{H})(I_{E^{\otimes 2}} \otimes a)(p_2^* \otimes I_\mathcal{H})\widetilde{T}_2^* \\ &= \widetilde{T}_2(I_{X(2)} \otimes a)\widetilde{T}_2^* \text{ for all } a \in \sigma(\mathcal{M})'.\end{aligned}$$

Inductively, we get

$$\begin{aligned}\Theta_T^n(a) &= \Theta_T(\Theta_T^{n-1}(a)) = \widetilde{T}_1(I_E \otimes (\widetilde{T}_{n-1}(I_{X(n-1)} \otimes a)\widetilde{T}_{n-1}^*))\widetilde{T}_1^* \\ &= \widetilde{T}_1(I_E \otimes \widetilde{T}_{n-1})(I_E \otimes I_{X(n-1)} \otimes a)(I_E \otimes \widetilde{T}_{n-1}^*)\widetilde{T}_1^* \\ &= \widetilde{T}_n(p_n \otimes I_\mathcal{H})(I_E \otimes I_{X(n-1)} \otimes a)(p_n^* \otimes I_\mathcal{H})\widetilde{T}_n^* \\ &= \widetilde{T}_n(I_{X(n)} \otimes a)\widetilde{T}_n^*,\end{aligned}$$

for all $a \in \sigma(\mathcal{M})'$ and $n \geq 2$.

The following is a reformulation of Muhly and Solel's result in our setting [13, Proposition 2.12]:

Proposition 4.1. *Let* $X = (X(n))_{n\in\mathbb{Z}_+}$ *be a standard subproduct system of* W^**-correspondences over a finite factor* $\mathcal{M}$*. Assume that* $E := X(1)$ *is a left-finite* W^**-correspondence with* $d := dim_l(E)$*. If* $T = (T_n)_{n\in\mathbb{Z}_+}$ *is a completely contractive, covariant representation of* X *on* $\mathcal{H}$ *then*

$$tr_{\sigma(\mathcal{M})'}(\Theta_T(x)) \leq \|\widetilde{T}_1\|^2\, dim_l(E) tr_{\sigma(\mathcal{M})'}(x),$$

for all $x \in \sigma(\mathcal{M})'_+$.

Let $X = (X(n))_{n\in\mathbb{Z}_+}$ be a standard subproduct system of W^*-correspondences over a semifinite factor $\mathcal{M}$. Assume that $E := X(1)$ is a left-finite W^*-correspondence with $d := dim_l(E)$. Define *curvature* of a completely contractive, covariant representation $T = (T_n)_{n\in\mathbb{Z}_+}$ of X on a Hilbert space $\mathcal{H}$ by

$$Curv(T) = \lim_{k\to\infty} \frac{tr_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I))}{\sum_{j=0}^{k-1} d^j}.$$

The following useful result is due to Popescu [19, p.280].

Lemma 4.2. *Let* $\{a_j\}_{j=0}^{\infty}$ *and* $\{b_j\}_{j=0}^{\infty}$ *be two real sequences and* $b_j > 0$, $j \geq 0$. *Consider the partial sums* $A_k := \sum_{j=0}^{k-1} a_j$ *and* $B_k := \sum_{j=0}^{k-1} b_j$, *and suppose that* $B_k \to \infty$ *as* $k \to \infty$. *Then*

$$\lim_{k\to\infty} \frac{A_k}{B_k} = L,$$

whenever $L := \lim_{j\to\infty} \frac{a_j}{b_j}$ *exists.*

The following theorem is an analogue of [13, Theorem 3.3] for completely contractive, covariant representations on subproduct systems.

Theorem 4.3. *Let* $X = (X(n))_{n\in\mathbb{Z}_+}$ *be a standard subproduct system of* W^**-correspondences over a finite factor* $\mathcal{M}$. *Assume that* $E := X(1)$ *is a left-finite* W^**-correspondence and* $d := dim_l(E)$. *If* $T = (T_n)_{n\in\mathbb{Z}_+}$ *is a completely contractive, covariant representation of* X *on* $\mathcal{H}$*, then the following holds:*

1. *The limit in the definition of* $Curv(T)$ *exists (either as a finite positive number or* $+\infty$*).*
2. $Curv(T) = \infty$ *if and only if* $tr_{\sigma(\mathcal{M})'}(I - \Theta_T(I)) = \infty$.
3. *If* $tr_{\sigma(\mathcal{M})'}(I - \Theta_T(I)) < \infty$ *then* $Curv(T) < \infty$. *Moreover, in this case we have the following:*
   (3a) for $d \geq 1$ *we have*
   $$Curv(T) = \lim_{k\to\infty} \frac{tr_{\sigma(\mathcal{M})'}(\Theta_T^k(I) - \Theta_T^{k+1}(I))}{d^k},$$
   in particular, if $d > 1$*, then we further get*
   $$Curv(T) = (d-1) \lim_{k\to\infty} \frac{tr_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I))}{d^k};$$
   (3b) for $d < 1$, $\lim_{k\to\infty} tr_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I)) < \infty$*, and*
   $$Curv(T) = (1-d)\left(\lim_{k\to\infty} tr_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I))\right).$$

Proof. Let $a_k = tr_{\sigma(\mathcal{M})'}(\Theta_T^k(I) - \Theta_T^{k+1}(I))$ for $k \geq 0$. Then Proposition 4.1 yields

$$\begin{aligned} a_{k+1} &= tr_{\sigma(\mathcal{M})'}(\Theta_T(\Theta_T^k(I) - \Theta_T^{k+1}(I))) \\ &\leq \|\widetilde{T}_1\|^2 dim_l(E) tr_{\sigma(\mathcal{M})'}(\Theta_T^k(I) - \Theta_T^{k+1}(I)) \\ &\leq da_k, \end{aligned}$$

for all $k \geq 0$. If $a_0 = \infty$, then the fact that $\{\widetilde{T}_n \widetilde{T}_n^*\}_{n\in\mathbb{Z}_+}$ is a decreasing sequence of positive contractions implies that

$$tr_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I)) = \infty \qquad (k \geq 0).$$

If $a_0 < \infty$, then $\{\frac{a_j}{d^j}\}_{j=0}^{\infty}$ is a non-increasing sequence of non-negative numbers. Then $0 \leq L \leq a_0$ where $L := \lim \frac{a_j}{d^j}$.
Let $d \geq 1$. Since

$$tr_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I)) = \sum_{j=0}^{k-1} a_j,$$

by Lemma 4.2 (for $b_j = d^j$) the limit defining $Curv(T)$ exists and $Curv(T) = L$.
Now let $d > 1$. Then $\sum_{j=0}^{k-1} d^j = \frac{d^k-1}{d-1}$ and $\lim_{k\to\infty} \frac{d^k-1}{d^k} = 1$ yields

$$\begin{aligned} Curv(T) &= \lim_{k\to\infty} \frac{tr_{\sigma(\mathcal{M})'}(I-\Theta_T^k(I))}{\frac{d^k-1}{d-1}} \\ &= (d-1)\lim_{k\to\infty} \frac{tr_{\sigma(\mathcal{M})'}(I-\Theta_T^k(I))}{d^k-1} \lim_{k\to\infty} \frac{d^k-1}{d^k} \\ &= (d-1)\lim_{k\to\infty} \frac{tr_{\sigma(\mathcal{M})'}(I-\Theta_T^k(I))}{d^k}. \end{aligned}$$

This proves statement $(3a)$.
Finally, let $d < 1$ so that $\sum_{j=0}^{\infty} d^j = 1/(1-d)$. Since $a_j \leq d^j a_0$ for all $j \geq 0$, $\lim_{k\to\infty} tr_{\sigma(\mathcal{M})'}(I - \Theta_T^k(I))$ exists and is finite. This completes the proof of $(3b)$. The proof of statements (1) and (2) follows by noting that whenever a_0 is finite, the limit defining $Curv(T)$ exists and is finite. □

Part (3a), $d > 1$ case, should be compared with the Popescu curvature (see Corollary 2.7 and the equality (2.16) in [19]). Also note that the Popescu curvature is a generalization of Arveson curvature (see Corollary 2.8 in [19]).

Recall that $\Theta_T(x) = \widetilde{T}_1(I_E \otimes x)\widetilde{T}_1^*$ for all $x \in \sigma(\mathcal{M})'$, and that $Q = \lim_{n\to\infty} \widetilde{T}_n \widetilde{T}_n^* = \lim_{n\to\infty} \Theta_T^n(I_{\mathcal{H}})$. Using the intertwining property $K(T)^*(S_n(\zeta) \otimes I_{\mathcal{D}}) = T_n(\zeta)K(T)^*$ of the Poisson kernel, we have

$$\begin{aligned} \widetilde{T}_n(I_{X(n)} \otimes K(T)^*)(\zeta \otimes k) &= \widetilde{T}_n(\zeta \otimes K(T)^* k) \\ &= T_n(\zeta)K(T)^* k \\ &= K(T)^*(S_n(\zeta) \otimes I_{\mathcal{D}})k, \end{aligned}$$

for all $\zeta \in X(n), k \in \mathcal{F}_X \bigotimes_\sigma \mathcal{D}, n \in \mathbb{Z}_+$. Then

$$\widetilde{T}_n(I_{X(n)} \otimes K(T)^*) = K(T)^*(\widetilde{S_n(\cdot) \otimes I_{\mathcal{D}}}),$$

and hence $\Theta_T^n(Q) = Q$ and $K(T)^* K(T) = I_{\mathcal{H}} - Q$ yields

$$
\begin{aligned}
&K(T)^*(I_{\mathcal{F}_X \bigotimes_\sigma \mathcal{D}} - \Theta^n_{S\otimes I_{\mathcal{D}}}(I_{\mathcal{F}_X \bigotimes_\sigma \mathcal{D}}))K(T)\\
&\qquad = K(T)^*K(T) - K(T)^*(\widetilde{S_n(\cdot)\otimes I_{\mathcal{D}}})(\widetilde{S_n(\cdot)\otimes I_{\mathcal{D}}})^*K(T)\\
&\qquad = K(T)^*K(T) - \widetilde{T}_n(I_{X(n)}\otimes K(T)^*)(\widetilde{S_n(\cdot)\otimes I_{\mathcal{D}}})^*K(T)\\
&\qquad = I_{\mathcal{H}} - Q - \widetilde{T}_n(I_{X(n)}\otimes K(T)^*)(I_{X(n)}\otimes K(T))\widetilde{T}_n^*\\
&\qquad = I_{\mathcal{H}} - Q - \widetilde{T}_n(I_{X(n)}\otimes K(T)^*K(T))\widetilde{T}_n^*\\
&\qquad = I_{\mathcal{H}} - Q - \widetilde{T}_n(I_{X(n)}\otimes (I_{\mathcal{H}} - Q))\widetilde{T}_n^*\\
&\qquad = I_{\mathcal{H}} - Q - \Theta^n_T(I_{\mathcal{H}} - Q)\\
&\qquad = I_{\mathcal{H}} - \Theta^n_T(I_{\mathcal{H}}).
\end{aligned}
$$

Therefore one can compute the curvature, in terms of Poisson kernel, in the following sense:

Proposition 4.4. *Let $X = (X(n))_{n\in\mathbb{Z}_+}$ be a standard subproduct system of W^*-correspondences over a finite factor $\mathcal{M}$. Assume that $E := X(1)$ is a left-finite W^*-correspondence with $d := dim_l(E)$. If $T = (T_n)_{n\in\mathbb{Z}_+}$ is a completely contractive, covariant representation of X on a Hilbert space $\mathcal{H}$, then the curvature of T is given by*

$$
Curv(T) = \lim_{k\to\infty} \frac{tr_{\sigma(\mathcal{M})'}(K(T)^*(I_{\mathcal{F}_X \bigotimes_\sigma \mathcal{D}} - \Theta^k_{S\otimes I_{\mathcal{D}}}(I_{\mathcal{F}_X \bigotimes_\sigma \mathcal{D}}))K(T))}{\sum_{j=0}^{k-1} d^j}.
$$

The following theorem generalizes [19, Theorem 2.1].

Theorem 4.5. *Assume $T = (T_n)_{n\in\mathbb{Z}_+}$ to be a completely contractive, covariant representation of a standard subproduct system $X = (X(n))_{n\in\mathbb{Z}_+}$ of A^*-correspondences over an A^*-algebra $\mathcal{M}$. Then there exist a Hilbert space $\mathcal{E}$, a representation ψ of $\mathcal{M}$ on $\mathcal{E}$, and an inner multi-analytic operator $\Pi : \mathcal{F}_X \bigotimes_\psi \mathcal{E} \to \mathcal{F}_X \bigotimes_\sigma \mathcal{D}$ such that*

$$
I_{\mathcal{F}_X \bigotimes_\sigma \mathcal{D}} - K(T)K(T)^* = \Pi\Pi^*.
$$

Proof. Proposition 2.5 implies that $(ranK(T))^\perp$ is invariant for the covariant representation $S\otimes I_{\mathcal{D}}$. Now we use Theorem 3.4 and obtain a Hilbert space $\mathcal{E}$, a representation ψ of $\mathcal{M}$ on $\mathcal{E}$, and a partial isometry $\Pi : \mathcal{F}_X \bigotimes_\psi \mathcal{E} \to \mathcal{F}_X \bigotimes_\sigma \mathcal{D}$ such that $(ranK(T))^\perp$ is the range of Π, and

$$
\Pi(S_n(\zeta)\otimes I_{\mathcal{E}}) = (S_n(\zeta)\otimes I_{\mathcal{D}})\Pi \quad whenever\ \zeta\in X(n),\ n\in\mathbb{Z}_+.
$$

Finally, using the fact that Π is a partial isometry and $(ranK(T))^\perp = ran(I_{\mathcal{F}_X \bigotimes_\sigma \mathcal{D}} - K(T)K(T)^*)$, we get the desired formula. ☐

The following is an analogue of [20, Theorem 3.32] in our context.

Theorem 4.6. *Let $X = (X(n))_{n\in\mathbb{Z}_+}$ be a standard subproduct system of W^*-correspondences over a finite factor $\mathcal{M}$. Assume that $E := X(1)$ is a left-finite W^*-correspondence with $d := dim_l(E)$. If $T = (T_n)_{n\in\mathbb{Z}_+}$ is a completely contractive, covariant representation of X on $\mathcal{H}$, and*

$$
tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}(I_{\mathcal{F}_X \bigotimes_\sigma \mathcal{D}} - \Theta_{S\otimes I_{\mathcal{D}}}(I_{\mathcal{F}_X \bigotimes_\sigma \mathcal{D}})) < \infty,
$$

then there exist a Hilbert space $\mathcal{E}$, a representation ψ of $\mathcal{M}$ on $\mathcal{E}$, and an inner multi-analytic operator $\Pi : \mathcal{F}_X \bigotimes_\psi \mathcal{E} \to \mathcal{F}_X \bigotimes_\sigma \mathcal{D}$ *such that*

$$Curv(T) = \lim_{k\to\infty} \frac{tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}((I_{\mathcal{F}_X \otimes_\sigma \mathcal{D}} - \Pi\Pi^*)(I_{\mathcal{F}_X \otimes_\sigma \mathcal{D}} - \Theta^k_{S\otimes I_{\mathcal{D}}}(I_{\mathcal{F}_X \otimes_\sigma \mathcal{D}})))}{\sum_{j=0}^{k-1} d^j}.$$

Proof. For simplicity of notation we use I for $I_{\mathcal{F}_X \otimes_\sigma \mathcal{D}}$ and also use Θ for $\Theta_{S\otimes I_{\mathcal{D}}}$. Define a representation ρ of $\mathcal{M}$ on $\mathcal{H} \bigoplus (\mathcal{F}_X \bigotimes_\sigma \mathcal{D})$ by

$$\rho(a) = \begin{pmatrix} \sigma(a) & 0 \\ 0 & \phi_\infty(a)\otimes I_{\mathcal{D}} \end{pmatrix} \text{ for all } a \in \mathcal{M}.$$

Therefore we have

$$\begin{aligned}
& tr_{\sigma(\mathcal{M})'}(K(T)^*(I - \Theta^k(I))K(T)) \\
&= tr_{\rho(\mathcal{M})'} \begin{pmatrix} K(T)^*(I-\Theta^k(I))K(T) & 0 \\ 0 & 0 \end{pmatrix} \\
&= tr_{\rho(\mathcal{M})'} \left(\begin{pmatrix} 0 & K(T)^*(I-\Theta^k(I))^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ (I-\Theta^k(I))^{\frac{1}{2}}K(T) & 0 \end{pmatrix} \right) \\
&= tr_{\rho(\mathcal{M})'} \left(\begin{pmatrix} 0 & 0 \\ (I-\Theta^k(I))^{\frac{1}{2}}K(T) & 0 \end{pmatrix} \begin{pmatrix} 0 & K(T)^*(I-\Theta^k(I))^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \right) \\
&= tr_{\rho(\mathcal{M})'} \begin{pmatrix} 0 & 0 \\ 0 & (I-\Theta^k(I))^{\frac{1}{2}}K(T)K(T)^*(I-\Theta^k(I))^{\frac{1}{2}} \end{pmatrix} \\
&= tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}((I-\Theta^k(I))^{\frac{1}{2}}K(T)K(T)^*(I-\Theta^k(I))^{\frac{1}{2}}). \qquad (4.1)
\end{aligned}$$

Now by Theorem 4.5, there exist a Hilbert space $\mathcal{E}$, a representation ψ of $\mathcal{M}$ on $\mathcal{E}$, and an inner multi-analytic operator $\Pi : \mathcal{F}_X \bigotimes_\psi \mathcal{E} \to \mathcal{F}_X \bigotimes_\sigma \mathcal{D}$ such that

$$Curv(T) = \lim_{k\to\infty} \frac{tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}((I_{\mathcal{F}_X \otimes_\sigma \mathcal{D}} - \Pi\Pi^*)(I_{\mathcal{F}_X \otimes_\sigma \mathcal{D}} - \Theta^k_{S\otimes I_{\mathcal{D}}}(I_{\mathcal{F}_X \otimes_\sigma \mathcal{D}})))}{\sum_{j=0}^{k-1} d^j}.$$

Then (4.1) and Proposition 4.4 yields

$$\begin{aligned}
Curv(T) &= \lim_{k\to\infty} \frac{tr_{\sigma(\mathcal{M})'}(K(T)^*(I-\Theta^k(I))K(T))}{\sum_{j=0}^{k-1} d^j} \\
&= \lim_{k\to\infty} \frac{tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}((I-\Theta^k(I))^{\frac{1}{2}}K(T)K(T)^*(I-\Theta^k(I))^{\frac{1}{2}})}{\sum_{j=0}^{k-1} d^j} \\
&= \lim_{k\to\infty} \frac{tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}((I-\Theta^k(I))^{\frac{1}{2}}(I-\Pi\Pi^*)(I-\Theta^k(I))^{\frac{1}{2}})}{\sum_{j=0}^{k-1} d^j} \\
&= \lim_{k\to\infty} \frac{tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}((I-\Pi\Pi^*)(I-\Theta^k(I)))}{\sum_{j=0}^{k-1} d^j}.
\end{aligned}$$

The last equality follows from the following two observations:

(1) the recurrence relation $I - \Theta^k(I) = I - \Theta(I) + \Theta(I - \Theta^{k-1}(I))$ implies that $tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}(I-\Theta^k(I)) \leq (\sum_{m=0}^{k-1} d^m)\ tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}(I-\Theta(I)) < \infty$,

(2) since $tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}(I-\Theta^k(I))$ is finite, $(I-\Theta^k(I))^{\frac{1}{2}}$ belongs to the ideal

$$\{x : tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}(x^*x) < \infty\},$$

and hence $tr_{(\phi_\infty(\mathcal{M})\otimes I_{\mathcal{D}})'}(I-\Theta^k(I))^{\frac{1}{2}} < \infty$, for all k. □

5. Wandering subspaces

The notion of wandering subspaces of bounded linear operators on Hilbert spaces was introduced by Halmos [8]. With this as a motivation we extend the notion of wandering subspace (cf. [9, p. 561]) for covariant representations of standard subproduct systems, as follows: Let $T = (T_n)_{n\in\mathbb{Z}_+}$ be a covariant representation of a standard subproduct system $X = (X(n))_{n\in\mathbb{Z}_+}$ over an A^*-algebra $\mathcal{M}$. A closed subspace $\mathcal{S}$ of $\mathcal{H}$ is called *wandering* for the covariant representation T if it is $\sigma(\mathcal{M})$-invariant, and if for each $n \in \mathbb{N}$ the subspace $\mathcal{S}$ is orthogonal to

$$\mathfrak{L}_n(\mathcal{S}, T) := \bigvee\{T_n(p_n(\zeta))s \,:\, \zeta \in E^{\otimes n}, s \in \mathcal{S}\}.$$

When there is no confusion we use the notation $\mathfrak{L}_n(\mathcal{S})$ for $\mathfrak{L}_n(\mathcal{S}, T)$, and also use $\mathfrak{L}(\mathcal{S})$ for $\mathfrak{L}_1(\mathcal{S})$. A wandering subspace $\mathcal{W}$ for T is called *generating* if $\mathcal{H} = \overline{\text{span}}\{\mathfrak{L}_n(\mathcal{W}) : n \in \mathbb{Z}_+\}$.

In the following proposition we prove that the wandering subspaces are naturally associated with invariant subspaces of covariant representations of standard subproduct systems.

Proposition 5.1. *Let $T = (T_n)_{n\in\mathbb{Z}_+}$ be a covariant representation of a standard subproduct system $X = (X(n))_{n\in\mathbb{Z}_+}$ over an A^*-algebra $\mathcal{M}$. If $\mathcal{S}$ is a closed T-invariant subspace of $\mathcal{H}$, then $\mathcal{S} \ominus \mathfrak{L}(\mathcal{S})$ is a wandering subspace for $T|_{\mathcal{S}} := (T_n|_{\mathcal{S}})_{n\in\mathbb{Z}_+}$.*

Proof. Let $n \geq 1$ and $\eta = \xi_1 \otimes \xi_{n-1} \in E^{\otimes n}$ for some $\xi_1 \in E$ and $\xi_{n-1} \in E^{\otimes n-1}$. Let $x, s \in \mathcal{S} \ominus \mathfrak{L}(\mathcal{S})$ so that $y = T_n(p_n(\eta))s \in \mathfrak{L}_n(\mathcal{S} \ominus \mathfrak{L}(\mathcal{S}))$. Then

$$\begin{aligned}
\langle x, y\rangle &= \langle x, T_n(p_n(\eta))s\rangle \\
&= \langle x, T_n(p_n(\xi_1 \otimes \xi_{n-1}))s\rangle \\
&= \langle x, T_{1+(n-1)}(p_{1+(n-1)}(\xi_1 \otimes \xi_{n-1}))s\rangle \\
&= \langle x, T_1(\xi_1)T_{n-1}(\xi_{n-1})s\rangle \\
&= 0,
\end{aligned}$$

since $\mathcal{S}$ in invariant under $T_{n-1}(\xi_{n-1})$. Therefore $\mathcal{S}\ominus\mathfrak{L}(\mathcal{S})$ is orthogonal to $\mathfrak{L}_n(\mathcal{S}\ominus\mathfrak{L}(\mathcal{S}))$, $n \geq 1$ and hence $\mathcal{S} \ominus \mathfrak{L}(\mathcal{S})$ is a wandering subspace for $T|_{\mathcal{S}} = (T_n|_{\mathcal{S}})_{n\in\mathbb{Z}_+}$. □

Let $T = (T_n)_{n\in\mathbb{Z}_+}$ be a covariant representation of a standard subproduct system $X = (X(n))_{n\in\mathbb{Z}_+}$. Suppose $\mathcal{W}$ is a wandering subspace for T. Set

$$\mathcal{G}_{T,\mathcal{W}} := \bigvee_{n\in\mathbb{Z}_+} \mathfrak{L}_n(\mathcal{W}).$$

Note that

$$
\begin{aligned}
\mathfrak{L}\left(\bigvee_{n\in\mathbb{Z}_+}\mathfrak{L}_n(\mathcal{W})\right) &= \overline{\operatorname{span}}\{T_1(p_1(\zeta))T_n(p_n(\eta))w : \zeta\in E, \eta\in E^{\otimes n}, w\in\mathcal{W}, n\in\mathbb{Z}_+\}\\
&= \overline{\operatorname{span}}\{T_{n+1}(p_{n+1}(p_1(\zeta)\otimes p_n(\eta))w : \zeta\in E, \eta\in E^{\otimes n}, w\in\mathcal{W}, n\in\mathbb{Z}_+\}\\
&\subset \bigvee_{n\in\mathbb{N}}\mathfrak{L}_n(\mathcal{W}).
\end{aligned}
$$

In the other direction, we have

$$
\begin{aligned}
\bigvee_{n\in\mathbb{N}}\mathfrak{L}_n(\mathcal{W}) &= \overline{\operatorname{span}}\{T_n(p_n(p_1(\zeta)\otimes p_{n-1}(\eta))w : \zeta\in E, \eta\in E^{\otimes n-1}, w\in\mathcal{W}, n\in\mathbb{N}\}\\
&= \overline{\operatorname{span}}\{T_1(p_1(\zeta))T_{n-1}(p_{n-1}(\eta))w : \zeta\in E, \eta\in E^{\otimes n-1}, w\in\mathcal{W}, n\in\mathbb{N}\}\\
&\subset \mathfrak{L}\left(\bigvee_{n\in\mathbb{Z}_+}\mathfrak{L}_n(\mathcal{W})\right).
\end{aligned}
$$

Thus these sets are equal, and it follows that

$$
\mathcal{G}_{T,\mathcal{W}}\ominus\mathfrak{L}(\mathcal{G}_{T,\mathcal{W}}) = \bigvee_{n\in\mathbb{Z}_+}\mathfrak{L}_n(\mathcal{W})\ominus\mathfrak{L}\left(\bigvee_{n\in\mathbb{Z}_+}\mathfrak{L}_n(\mathcal{W})\right) = \mathcal{W}.
$$

Hence we have the following uniqueness result:

Proposition 5.2. *Let $T=(T_n)_{n\in\mathbb{Z}_+}$ be a covariant representation of a standard subproduct system $X=(X(n))_{n\in\mathbb{Z}_+}$ over an A^*-algebra $\mathcal{M}$. If $\mathcal{W}$ is a wandering subspace for T, then*

$$\mathcal{W} = \mathcal{G}_{T,\mathcal{W}}\ominus\mathfrak{L}(\mathcal{G}_{T,\mathcal{W}}).$$

Moreover, if $\mathcal{W}$ is also generating, then $\mathcal{W}=\mathcal{H}\ominus\mathfrak{L}(\mathcal{H})$.

In Theorem 3.1 we observed that each non-trivial closed subspace $\mathcal{S}\subset\mathcal{H}$, which is invariant under a pure completely contractive, covariant representation $T=(T_n)_{n\in\mathbb{Z}_+}$ of a standard subproduct system $X=(X(n))_{n\in\mathbb{Z}_+}$, can be written as $\mathcal{S}=\Pi(\mathcal{F}_X\bigotimes_\pi\mathcal{D})$. In the following theorem we study wandering subspaces in a general situation when T is not necessarily pure.

Theorem 5.3. *Let $X=(X(n))_{n\in\mathbb{Z}_+}$ be a standard subproduct system over an A^*-algebra $\mathcal{M}$. Let $\pi:\mathcal{M}\to B(\mathcal{E})$ be a representation on a Hilbert space $\mathcal{E}$ and $T=(T_n)_{n\in\mathbb{Z}_+}$ be the covariant representation of X. Let $\Pi:\mathcal{F}_X\bigotimes_\pi\mathcal{E}\to\mathcal{H}$ be a partial isometry such that $\Pi(S_n(\zeta)\otimes I_{\mathcal{E}})=T_n(\zeta)\Pi$ for every $\zeta\in X(n), n\in\mathbb{Z}_+$. Then $\mathcal{S}:=\Pi(\mathcal{F}_X\bigotimes_\pi\mathcal{E})$ is a closed T-invariant subspace, $\mathcal{W}:=\mathcal{S}\ominus\mathfrak{L}(\mathcal{S})$ is a wandering subspace for $T|_{\mathcal{S}}$, and $\mathcal{W}=\Pi((ker\Pi)^\perp\bigcap\mathcal{M}\bigotimes_\pi\mathcal{E})$.*

Proof. Define $F = (ker\Pi)^{\perp} \bigcap \mathcal{M} \bigotimes_{\pi} \mathcal{E}$. Since $\mathcal{S}$ is the range of Π, it is a closed T-invariant subspace. Therefore by Proposition 5.1, the subspace $\mathcal{W}$ is a wandering subspace for $T|_{\mathcal{S}}$.

$$\begin{aligned}
\mathfrak{L}(\mathcal{S}, T) &= \mathfrak{L}(\Pi(\mathcal{F}_X \textstyle\bigotimes_{\pi} \mathcal{E}), T) \\
&= \bigvee\{\, T_1(\zeta)k \;:\; k \in \Pi(\mathcal{F}_X \textstyle\bigotimes_{\pi} \mathcal{E}), \zeta \in X(1)\} \\
&= \bigvee\{\, T_1(\zeta)\Pi(l) \;:\; l \in \mathcal{F}_X \textstyle\bigotimes_{\pi} \mathcal{E}, \zeta \in X(1)\} \\
&= \bigvee\{\, \Pi(S_1(\zeta) \otimes I_{\mathcal{E}})(l_m \otimes e) \;:\; l_m \otimes e \in X(m) \textstyle\bigotimes_{\pi} \mathcal{E}, \zeta \in X(1), m \in \mathbb{Z}_+\}.
\end{aligned}$$

For $x \in (ker\Pi)^{\perp} \bigcap \mathcal{M} \bigotimes_{\pi} \mathcal{E}$ and $l_m \otimes e \in X(m) \bigotimes_{\pi} \mathcal{E}$ we have

$$\begin{aligned}
\langle \Pi x, \Pi(S_1(\zeta) \otimes I_{\mathcal{E}})(l_m \otimes e)\rangle &= \langle \Pi^*\Pi x, (S_1(\zeta) \otimes I_{\mathcal{E}})(l_m \otimes e)\rangle \\
&= \langle x, (S_1(\zeta) \otimes I_{\mathcal{E}})(l_m \otimes e)\rangle \\
&= \langle x, P_{1+m}(\zeta \otimes l_m) \otimes e\rangle \\
&= 0,
\end{aligned}$$

and hence $\Pi((ker\Pi)^{\perp} \bigcap \mathcal{M} \bigotimes_{\pi} \mathcal{E}) \subset \mathcal{W}$.
For the converse direction, let $x \in \mathcal{S} \ominus \mathfrak{L}(\mathcal{S}, T) = \mathcal{W}$, and $\Pi(y) = x$ for some $y \in (ker\Pi)^{\perp}$. Therefore for any $\zeta \in X(1), \eta \otimes e \in \mathcal{F}_X \bigotimes_{\pi} \mathcal{E}$ we have

$$\langle y, (S_1(\zeta) \otimes I_{\mathcal{E}})(\eta \otimes e)\rangle \;=\; \langle \Pi y, \Pi(S_1(\zeta) \otimes I_{\mathcal{E}})(\eta \otimes e)\rangle = 0. \tag{5.1}$$

Recall that by definition we have

$$\mathfrak{L}(\mathcal{F}_X \textstyle\bigotimes_{\pi} \mathcal{E}, S \otimes I_{\mathcal{E}}) = \displaystyle\bigvee\{(S_1(\zeta) \otimes I_{\mathcal{E}})(\eta \otimes e) : \eta \in X(m), \zeta \in X(1), e \in \mathcal{E}, m \in \mathbb{Z}_+\}.$$

Since $\mathcal{M} \bigotimes_{\pi} \mathcal{E}$ is a generating wandering subspace for the covariant representation $S \otimes I_{\mathcal{E}}$, it follows from Proposition 5.2 that $(\mathcal{F}_X \bigotimes_{\pi} \mathcal{E}) \ominus \mathfrak{L}(\mathcal{F}_X \bigotimes_{\pi} \mathcal{E}, S \otimes I_{\mathcal{E}}) = \mathcal{M} \bigotimes_{\pi} \mathcal{E}$, and hence (5.1) implies that $y \in \mathcal{M} \bigotimes_{\pi} \mathcal{E}$. Hence $\Pi((ker\Pi)^{\perp} \bigcap \mathcal{M} \bigotimes_{\pi} \mathcal{E}) = \mathcal{W}$. $\square$

Since each commuting tuple of operators defines a covariant representation, the previous theorem is a generalization of [5, Theorem 5.2]. Indeed, we get the following corollary:

Corollary 5.4. *With the same notation of Theorem 5.3 we have*

$$\bigvee_{n \in \mathbb{Z}_+} \mathfrak{L}_n(\mathcal{W}, T) = \Pi\left(\bigvee_{n \in \mathbb{Z}_+} \mathfrak{L}_n(F, S \otimes I_{\mathcal{E}})\right)$$

where $F = (ker\Pi)^{\perp} \bigcap \mathcal{M} \bigotimes_{\pi} \mathcal{E}$. Moreover, F is wandering subspace for the representation $S \otimes I_{\mathcal{E}}$, that is, $F \perp \mathfrak{L}_n(F, S \otimes I_{\mathcal{E}})$ for each $n \in \mathbb{N}$.

Proof. For each $f, f' \in F$ we have

$$\begin{aligned}
\langle f, (S_n(\zeta) \otimes I_{\mathcal{E}})f'\rangle &= \langle \Pi^*\Pi f, (S_n(\zeta) \otimes I_{\mathcal{E}})f'\rangle = \langle \Pi f, \Pi(S_n(\zeta) \otimes I_{\mathcal{E}})f'\rangle \\
&= 0.
\end{aligned}$$

Therefore F is wandering subspace for the representation $S \otimes I_{\mathcal{E}}$. Moreover, since $\mathcal{W} = \Pi F$, we have that

$$\begin{aligned}\bigvee\{\mathfrak{L}_n(\mathcal{W}, T) : n \in \mathbb{Z}_+\} &= \bigvee\{(T_n(\zeta)\Pi(F) : \zeta \in X(n), n \in \mathbb{Z}_+\} \\ &= \bigvee\{\Pi(S_n(\zeta) \otimes I_{\mathcal{E}})(F) : \zeta \in X(n), n \in \mathbb{Z}_+\} \\ &= \Pi(\bigvee\{(S_n(\zeta) \otimes I_{\mathcal{E}})(F) : \zeta \in X(n), n \in \mathbb{Z}_+\}) \\ &= \Pi(\bigvee\{\mathfrak{L}_n(F, S \otimes I_{\mathcal{E}}) : n \in \mathbb{Z}_+\}). \qquad \square\end{aligned}$$

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